THOMSON REUTERS ENDNOTE
[Venherskyi* *et al.*, 5(11): November, 2016] ISSN: 2277-9655
IC™ Value: 3.00 Impact Factor: 4.116
CODEN: IJESS7

IJESRT
INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

INVESTIGATION OF THE PROPERTIES OF SOLUTION OF VARIATIONAL PROBLEM COUPLED FLOW OF SURFACE AND GROUNDWATERS

Petro Venherskyi [*], Heorhiy Shynkarenko[*,]**
[*]Department of applied mathematics and informatics, Franko national university of Lviv, Ukraine
[**]Department of mathematics and applied informatics, Opole University of Technology, Poland

DOI: 10.5281/zenodo.167102

ABSTRACT

The mathematical model of surface and ground flows of water are observed. Formulated initial - boundary and variational problems. We give conditions of the interaction of these two streams on the common border. And formulated a theorem on uniqueness and limited solution by variational problem of coupled stream.

KEYWORDS: Surface flow, groundwater flow, watershed, coupled flows, conditions of interface, convergence of numerical scheme, conditions of uniqueness solutions of the problem.

INTRODUCTION

To simplify the description of the motion of water flows on the watershed area is held vertical decomposition of the problem - the whole area is divided into layers: the surface layer of the atmosphere, land surface, unsaturated zone, saturated zone, zone traffic pressure and so on. In each layer to describe the movement of moisture used models of different dimensions and their solutions are connected by boundary conditions [4-8].

We select in solid medium (liquid) moving surface layer (Figure 1) of such a structure

$$\Omega_F(t) := \{(x_1,x_2,x_3) \in R^3, \quad \eta(x) < x_3 < \nu(x,t) \quad \forall x = (x_1,x_2) \in \Omega(t)\}.$$

Denote projection of its lower

$$\Omega(t) := \{(x_1,x_2,x_3) \in R^3 \mid x_3 = \eta(x), \ \forall x = (x_1,x_2) \in \Omega(t)\}$$

and upper

$$\Lambda_F(t) := \{(x_1,x_2,x_3) \in R^3 \mid x_3 = \nu(x,t) \ \ \forall x = (x_1,x_2) \in \Omega(t)\}$$

bases on the plane $0x_1x_2$. The rest of the surface layer

$$\Gamma_F(t) := \{(x_1,x_2,x_3) \in R^3, \quad \eta(x) < x_3 < \nu(x,t) \quad \forall x = (x_1,x_2) \in \Omega(t)\}$$

will be called the lateral surface layer $F(t)$.

Similarly denote part of fluid that moves in the soil, so

$$\Omega_P(t) := \{(x_1,x_2,x_3) \in R^3, \quad \mathrm{h}(x) < x_3 < \eta(x), \quad \forall x = (x_1,x_2) \in \Omega(t)\}$$

the projection of the lower part will be written as

$$\Lambda_P(t) := \{(x_1,x_2,x_3) \in R^3 \mid x_3 = h(x), \ \forall x = (x_1,x_2) \in \Omega(t)\}.$$

Then, a layer of ground water

$$\Gamma_P(t) := \{(x_1,x_2,x_3) \in R^3, \quad \mathrm{h}(x) < x_3 < \eta(x) \quad \forall x \in \Gamma_P(t)\}$$

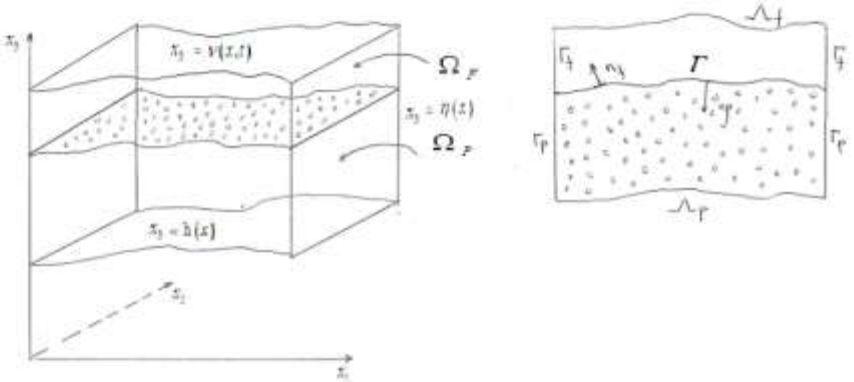

Figure 1. General view of the model of flows and their cross-section.

MATERIALS AND METHODS

1. Initial boundary value problem of interaction of water flows.

We formulate initial boundary problem of motion of surface and groundwater flows on the surface watershed considering boundary and initial conditions [1-3].

$$\begin{cases} \text{Find unknown quantities } \{u, p, \varphi\} \text{ such that} \\ \text{satisfy the following system of equations:} \\ \dfrac{\partial}{\partial t}(\rho u_i) + \sum_{k=1}^{3} \dfrac{\partial}{\partial x_k}(\rho\, u_i u_k) - \rho\, f_i - \sum_{k=1}^{3} \dfrac{\partial \sigma_{ik}}{\partial x_k} = 0, \qquad (1) \\ \sigma_{ij} = -p_F \delta_{ij} + \tau_{ij}, \\ \tau_{ij} = 2\mu\, e_{ij}, i, j = 1,2,3 \\ e_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right), \\ \dfrac{\partial \rho}{\partial t} + \sum_{k=1}^{3} \dfrac{\partial(\rho\, u_k)}{\partial x_k} = 0, \quad \text{in} \quad \Omega_F \times (0, T], \qquad (2) \\ m \dfrac{\partial \varphi}{\partial t} = \sum_{j=1}^{3} \dfrac{\partial}{\partial x_j}\left(k \dfrac{\partial \varphi}{\partial x_j}\right) + \varepsilon \quad in \quad \Omega_P \times (0; T], \qquad (3) \end{cases}$$

where $\{u_i(x,t)\}^3_{i=1}$ and $p_F = p_F(x,t)$ – the sought velocity vector of fluid and hydrostatic pressure, respectively; $F = \{g\ f_i(x)\}^3_{i=1}$ – the mass forces; $\rho=\rho(x,t)>0$ – density of the mass water flow; $\mu=\mu(x)>0$ – coefficient of viscosity; $\{e_{ij}\}^3_{i,j=1}$, $\{\sigma_{ij}\}^3_{i,j=1}$ – tensors velocities of deformation and stress of the liquid at the point x in time t; δ_{ij} – Kronecker symbol; $k = k(x,t)$ - coefficient of filtration; $m = m(x,t)$ - the coefficient of specific water loss; $\varepsilon = \varepsilon(x,t)$ - known function of sources of water influx;

$\varphi = x_3 + \dfrac{p_p}{\rho g}$ - piezometric pressure; (4)

$q = -k\nabla\varphi$ - flow (flow separation); (5)

$\upsilon = \upsilon(x,t)$ - the velocity vector of fluid in the ground;

$\upsilon = \dfrac{q}{\omega}$, ω - volume porosity;

ISSN: 2277-9655
Impact Factor: 4.116
CODEN: IJESS7

$\overrightarrow{n_F} = -\overrightarrow{n_P}$ - vectors normal to the boundary area Ω_F and Ω_P in accordance;

$\overline{\Omega} = \overline{\Omega_F} \cup \overline{\Omega_P}, \Omega_F \cap \Omega_P = \{\varnothing\}, \overline{\Omega_F} \cap \overline{\Omega_P} = \Gamma,\ \partial\Omega_F = \Gamma_F \cup \Lambda_F \cup \Gamma; \partial\Omega_P = \Gamma_P \cup \Lambda_P \cup \Gamma.$

Boundary conditions:

$\vec{u}_i = 0$ on Γ_F, $i = 1, 2,3,$ (6)

$\sigma_{n\tau} = \sigma$, on Λ_F, (7)

$$u_3 + R = \frac{\partial v}{\partial t} + u_1^0 \frac{\partial v}{\partial x_1} + u_2^0 \frac{\partial v}{\partial x_2} \text{ в } \Omega_F \times (0, T], \quad (8)$$

where R - velocity of falling rain drops, u_1^0, u_2^0 - horizontal components of velocity on the free surface $v(x.t)$ (Λ_F);

$$\upsilon_\bullet \overrightarrow{n_P} = \overline{\upsilon} \quad на \quad \Gamma_P; \quad (9)$$

$$\upsilon_1 = \upsilon_2 = 0 \quad на \quad \Lambda_P, \quad (10)$$

$\upsilon_3 = -I$ на Λ_P, (11)

where I – known function that describes the velocity of flow of fluid through the surface Λ_P.

Initial conditions:

$u|_{t=0} = u_0,$

$p|_{t=0} = p_0,$ in Ω. (12)

$\varphi|_{t=0} = \varphi_0.$

Contact flow conditions on a common boundary Γ:

$\sigma_{nn}(u, p_F) = p_p,$

$\sigma_{\tau n} = 0,$ (13)

$u_n = -\upsilon_n.$

2. Variational formulation of the problem of interaction of water flows.

We introduce the following bilinear forms:

$$M_V(r;w,q) = \int_V \sum_{i=1}^{3} r w_i q_i ds,\ N_v(w;u,q) = \int_v \sum_{k=1}^{3}\sum_{i=1}^{3} \rho w_k \frac{\partial u_i}{\partial x_k} q_i ds,\ C_v(w,q) = \int_v 2\mu\, e(w) : e(q) ds,$$

$$A_V(w,q) = -\int_V w\, divq ds,\ Y_V(w,q) = -\int_V w\, q_n d\gamma,\ B_V(p,w) = -\int_V \sum_{i=1}^{3} p\, .\nabla w ds.$$

Introduce spaces:

$H_F := \{\xi \in (H^1(\Omega_F))^3 \,|\, \xi = 0 \ \ on \ \ \Gamma\},$

$H_p := \{\psi \in H^1(\Omega_p) \,|\, \psi = 0 \ \ on \ \ \Gamma\},\ W := H_F \times H_P,\ \Im_j : W \to R,\ j = \overline{1,3}$

$$\langle \Im_1, \xi \rangle = \sum_{i=1}^{3} \int_{\Omega_F} \rho f_i \xi_i ds + \int_{\Lambda_F} (\xi_n p_a + \xi_\tau \cdot \hat{\sigma}) d\gamma,$$

$$\langle \Im_2, \theta \rangle = -\int_{\partial\Lambda_F} u_n^0\, \theta\, d\gamma,\ \langle \Im_3, \psi \rangle = \int_{\Omega_P} \frac{\varepsilon(x,t)\rho g \psi}{\omega} dp - \int_{\partial\Lambda_P} \upsilon\, \psi\, \rho g d\gamma.$$

Denote

$$\tilde{\psi} = \psi\, \rho\, g,\ \tilde{m} = \frac{m}{\omega},$$

Then write the following variational problem:

ISSN: 2277-9655
Impact Factor: 4.116
CODEN: IJESS7

Find $\{u, p, \varphi\} \in V \times Q \times W,$

$$M_{\Omega_F}(\rho; u', \xi) + N_{\Omega_F}(u; u, \xi) + A_{\Omega_F}(p, \xi) + C_{\Omega_F}(u, \xi) \qquad (14)$$

$$+Y_\Gamma(u, \xi) = \langle \Im_1, \xi \rangle, \forall \xi \in V$$

$$B_{\Omega_F}(u, \theta) + Y_\Gamma(\theta, u) = \langle \Im_2, \theta \rangle, \forall \theta \in Q \qquad (15)$$

$$M_{\Omega_P}(\tilde{m}; \varphi', \tilde{\psi}) + A_{\Omega_P}(\tilde{\psi}, \upsilon) + Y_\Gamma(\tilde{\psi}, \upsilon) = \langle \Im_3, \tilde{\psi} \rangle, \forall \psi \in W \qquad (16)$$

with initial conditions

$$M_{\Omega_F}(u'(0) - u_0, \xi) = 0, \qquad (17)$$

$$B_{\Omega_F}(p(0) - p_0, \theta) = 0; \qquad (18)$$

$$M_{\Omega_P}(\varphi'(0) - \varphi_0, \tilde{\psi}) = 0. \qquad (19)$$

Calculate considering initial conditions (17) - (19) and boundary conditions (6) - (11), and values of variables u and p with relations (14) and (15). Then of conditions of coupling flows (interface conditions) (13) and boundary condition (9) the value of the variable φ is calculated from (16).

3. The uniqueness and limited solution of variational problems of coupled flow.

We formulate the following theorem.

Theorem 1. Let variational problem (14) - (19), data which satisfy the conditions of regularity

$$u_0 \in H_F, \varphi_0 \in H_P, t_1, t_2 \in [0, T] \qquad (20)$$

has a solution $(u(t), \varphi(t))$.

Then the solution $(u(t), \varphi(t))$ is the single solution of the problem (1) - (13)

$$u \in L^2(0, T; H_F), \varphi \in L^2(0, T; H_P). \qquad (21)$$

Moreover, the solution $(u(t), \varphi(t))$ depends continuously on the data of (1) - (13) and under these conditions will correct a priori estimation

$$\frac{1}{2}\left[\|u(t)\|^2 + \|\varphi(t)\|^2 + \int_0^t (\|u(t)\|_W^2 + \|\varphi(t)\|_W^2) d\tau\right] \le$$
$$C\left\{\frac{1}{2}\left[\|u(0)\|^2 + \|\varphi(0)\|^2\right] + \int_0^t (\|u(t)\|_W^2 + \|\varphi(t)\|_W^2) d\tau\right\} \qquad (22)$$

with constant C>0, the value of which does not depend on the values of our interest.

Proof. Considering the conditions of the theorem (21), we have

$$l_1 \in L^2(0, T; H_F), l_2 \in L^2(0, T; H_P),$$

resulting of (14), (15) will correct estimates

$$\left|\int_0^t < \Im_1, u > d\tau\right| \le \frac{1}{2}\int_0^t (\|\Im_1(\tau)\|^2 + \|u(\tau)\|^2) d\tau \qquad (23)$$

$$\left|\int_0^t < \Im_2, u > d\tau\right| \le \frac{1}{2}\int_0^t (\|\Im_2(\tau)\|^2 + \|u(\tau)\|^2) d\tau \qquad (24)$$

Since the initial condition (12) for the function *u(t),* we have

$$\|u'(0)\|^2 = \|u_0\|_H^2 \quad . \qquad (25)$$

Given the inequality of Poincaré-Fridrihsa [9], has a place next estimation

$$\|u\|_{W_F} \le C\|u\|_{H'_F}, \forall u \in H'(\Omega_F), \qquad (26)$$

where

$$H'(\Omega_F) = \{u \in H'(\Omega_F) \mid u = 0 \text{ на } \Gamma = \partial\Omega_F\},$$

$$H'(\Omega_F) - \textit{space inverted to } H(\Omega_F).$$

Similarly, from (16) to function $\varphi(t)$ we write

$$\|\varphi'(0)\|^2 = \|\varphi_0\|_H^2. \qquad (27)$$

Further, in view of introduced norms in paragraph 4 [1], there is $C = const > 0$ that

$$|\varphi|_{W_P} \leq C\|\varphi\|_{H'_P}, \forall \varphi \in H'(\Omega_P),$$

where

$$H'(\Omega_P) = \{\varphi \in \mathrm{H}'(\Omega_p) \mid \varphi = 0 \ \ on \ \partial\Omega_P\}, \quad (28)$$

$$\mathrm{H}'(\Omega) - space \ \ inverted \ \ to \ \ H(\Omega).$$

then

$$|\vartheta(0)|^{H^b} \lessapprox C\|\vartheta(0)\|^{H_1^b} = C\|\vartheta^0\|^{H_1^b}, \forall \vartheta \in H_I(\Omega^b), \quad (29)$$

From relations (24) - (27) and (29) come to a priori assessment (22).
On basis of this evaluation by considerations of opposite the uniqueness solution of problem (1) - (13) is proved.

CONCLUSION

Based on the laws of conservation of considered the basic equations and boundary and initial conditions describing the coupled motion flows of surface and of ground water with unknown values of velocity and piezometric pressure. Variational problems formulated compatible flow and obtained the contact conditions on a common boundary based on the laws of motion a continuous medium. We prove uniqueness and limited variation problem solution coupled flow.

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